RESOLUTION OF AN ARBITRARY DISCONTINUITY IN MAGNETOHYDRODYNAMICS

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At the instant $t = 0$ let there be a discontinuity on the plane $x = 0$ in the parameters p, ρ, $\mathbf{H}$, $\mathbf{b}$ of a medium. Since the relations for the laws of conservation are not satisfied on that plane, the discontinuity cannot continue to exist in that form. The aim of this paper is to determine the motion of the medium in the subsequent moments of time.

A variety of problems can be reduced to the problem of the resolution of an arbitrary disturbance: the collision of masses of gas moving towards each other; various collisions of plane discontinuity surfaces; gases flying apart from each other; problems in which two motionless gases, in contact with each other at the initial instant, are compressed to different pressures and are in different magnetic fields; and so on.

The surface of the initial disturbance is not necessarily plane. In that case, our investigation at the initial moment of time is correct for sufficiently small portions of the surface of the initial discontinuity, each of which can be considered to be plane.

From the similarity properties of the problem, it follows that the motion must be composed of various combinations moving in both directions: fast (S^+) and slow (S^-) shock waves; fast (R^+) and slow (R^-) self-similar expansion waves; vorticity discontinuities (A); they are separated by a contact discontinuity (K). The symbols S^+, S^-, R^+, R^-, A, K denote the corresponding waves and discontinuities. The speed of propagation of these waves is such [1] that an S^+- or R^+-wave goes ahead, followed by an A-discontinuity, and this in turn by an S^-- or R^--wave. Thus, there can be three waves propagating in each direction, separated by a contact discontinuity. The problem of the resolution of an arbitrary disturbance is shown schematically in Fig. 1. If it is considered that some of the seven waves may be missing, then there are 648 different possible combinations of waves and discontinuities which may be realized, depending on the initial parameters of the medium to left and to right of the discontinuity.

In gasdynamics, the problem of the resolution of an arbitrary discontinuity was first solved by Kotchine [2,3].

In [4] the problem of the resolution of a discontinuity in a conducting medium was investigated for the case where there is a jump only in the tangential component of velocity in the plane of the discontinuity. All the other quantities are continuous, while the magnetic field is normal to the plane of the discontinuity. This problem is equivalent to the problem of a piston moving with a known velocity parallel to itself and normal to the field. Ahead of the piston there can be only one combination of waves, S^+R^-.

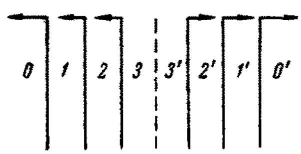


Fig. 1.

In [5] it was shown that if the magnetic field at the initial instant is parallel to the plane of the discontinuity, the problem reduces to a gasdynamic one. The problem of the resolution of an arbitrary discontinuity was investigated under these assumptions in [6,7,8].

The case where the initial disturbance, and therefore the secondary disturbances, is small was investigated in [9]. By virtue of the assumptions made it becomes possible to solve the problem by solving seven equations with seven unknowns; these were obtained by equating the sum of the jumps across the seven infinitesimal waves of each magnetohydrodynamic quantity to the initial jump.

Since the fluid is at rest relative to a contact discontinuity, then for a perfectly conducting medium the contact discontinuity may be considered to be a perfectly conducting piston moving with a velocity equal to the velocity of the contact discontinuity. In [10], assuming $H^2/8\pi \ll p$, $|\Delta \mathbf{b}| \ll c$, where c is the speed of sound and $\Delta \mathbf{b}$ the jump in velocity at the initial instant, it was possible to express the velocity of the contact discontinuity in terms of the parameters of the medium on either side of the discontinuity and thus reduce the problem to a piston problem, solved in the same paper.

In the general case of the resolution of an arbitrary discontinuity it is not possible to solve the corresponding system of equations, nor to reduce the problem to a piston problem [11].

In the present paper there is presented a method of solution which consists of the construction of a diagram in the space $\Delta u = u_0 - u_0'$, $\Delta v = v_0 - v_0'$, $\Delta w = w_0 - w_0'$, with the help of which, knowing Δu, Δv, Δw, it is possible to determine the combination of discontinuities which makes up the solution of the problem; then, writing out the relations at the discontinuities, there are no major difficulties in obtaining a final numerical solution.

The parameters characterizing the medium at the initial moment will be denoted by 0. The parameters of the medium lying to the right of the discontinuity at the initial moment and to the right of the contact surface at later times will be written with a prime. Those lying to the left of the corresponding surfaces will be written without a prime.

The parameters of the medium behind the first wave, going to left or right, will be denoted by 1, those behind the second wave by 2, and those behind the third wave by 3.

1. Conditions at shock waves. Kulikovskii [1] noted that the conditions at a shock wave may be solved in terms of the parameters of state ahead of the wave and the tangential component of magnetic field behind the wave. The corresponding expressions were obtained in [12], and are given below in a form due to A.A. Barmin.

$$\frac{\rho_1}{\rho_0} = \frac{h_0 Z_{\pm} + 1}{h_1 Z_{\pm} + 1} \tag{1.1}$$

$$p_1 = p_{\pm}(p_0, H_{\tau 0}, H_{\tau 1}) \equiv \gamma (h_1 - h_0)\left[Z_{\pm} - \frac{1}{2}(h_1 + h_0)\right] \tag{1.2}$$

$$(U - u_0)^2 = 1 + h_1 Z_{\pm} \tag{1.3}$$

$$u_1 - u_0 = \pm f_+, \qquad v_1 - v_0 = \mp \varphi_+ \operatorname{sign} H_{\tau_0} \tag{1.4}$$

$$u_1 - u_0 = \pm f_-, \qquad v_1 - v_0 = \pm \varphi_- \operatorname{sign} H_{\tau_0} \tag{1.5}$$

Here

$$f_{\pm} = \frac{h_1 - h_0}{\sqrt{1 + h_1 Z_{\pm}}} Z_{\pm} V_0, \qquad \varphi_+ = \left|\frac{h_1 - h_0}{\sqrt{1 + h_1 Z_+}} V_0\right|$$

$$\varphi_- = \left|\frac{h_1 - h_0}{\sqrt{1 + h_1 Z_-}} V_0\right|, \quad \mathbf{h} = \frac{\mathbf{H}_\tau}{H_n}, \quad P = \frac{4\pi\gamma p}{H_n^2} = \frac{c^2}{V^2}$$

The upper sign in Equations (1.4) and (1.5) corresponds to a wave travelling to the right, the lower one to a wave travelling to the left; $Z_{\pm}(h_1)$ is a physically sensible root of the equation

$$Z^2[(\gamma+1)h_0-(\gamma-1)h_1]-2Z\left(P_0-1+\frac{\gamma}{2}h_0^2-\frac{\gamma-2}{2}h_1h_0\right)-$$
$$-(h_1+h_0)=0$$

The root Z_+, which corresponds to the inequality $|h_1| - |h_0| > 0$, applies to the S^+-wave, the root Z_-, which corresponds to $|h_1| - |h_0| < 0$, applies to the S^--wave.

If $H_{\tau_0} > 0$, then $h_1 - h_0 > 0$, $Z_+ > 0$ in the S^+-wave, and $h_1 - h_0 < 0$, $Z_- < 0$ in the S^--wave; therefore $f_\pm > 0$. If $H_{\tau_0} < 0$ then $h_1 - h_0 < 0$, $Z_+ < 0$ in the S^+-wave, and $h_1 - h_0 > 0$, $Z_- > 0$ in the S^--wave; again $f_\pm > 0$. Here, and in what follows, u, v, w are absolute speeds of the gas, U is the speed of a shock wave, V is the Alfven speed, zero in the index in this and the following two sections refers to conditions ahead of the wave, the index 1 to conditions behind the wave.

2. Conditions at expansion waves. The conditions at expansion waves were solved by Friedrichs [4]

$$|H_{\tau_1}| = H_\pm(p_0, H_{\tau_0}, p_1) \equiv \sqrt{(q_\pm-1)(P-q_\pm^{-1})H_n^2} \tag{2.1}$$

$$u_1-u_0=\mp\psi_+, \qquad v_1-v_0=\pm\chi_+\operatorname{sign}H_{\tau_0} \tag{2.2}$$

$$u_1-u_0=\mp\psi_-, \qquad v_1-v_0=\mp\chi_-\operatorname{sign}H_{\tau_0} \tag{2.3}$$

$$\psi_\pm=\frac{V_0}{\gamma P_0^{1/2}}\int_{P_1}^{P_0}\left(\frac{P}{P_0}\right)^{-\frac{\gamma+1}{2\gamma}}q(P_0, q_0, P)\,dP$$
$$\chi_\pm=\frac{V_0}{\gamma P_0^{1/2}}\int_{P_1}^{P_0}\left(\frac{P}{P_0}\right)^{-\frac{\gamma+1}{2\gamma}}\left(\frac{1-q_\pm}{1-Pq_\pm}\right)^{\frac{1}{2}}dP \qquad \left(q_\pm=\frac{c_\pm^2}{c^2}\right) \tag{2.4}$$

where $c_\pm$ is the speed of propagation of weak, fast and slow magnetohydrodynamic waves, and $q_\pm = q(P_0, q_0, P)$ is the solution of the equation

$$\frac{dP}{dq}+\frac{\theta P}{1-q}=\frac{\theta 1}{q^2(1-q)}, \qquad \theta=\frac{\gamma}{2-\gamma}$$

The values of $q_\pm = q(P_0, q_0, P)$ which have physical sense are greater than unity, while the values of $q_- = q(P_0, q_0, P)$ which have physical sense are less than unity.

The upper sign in (2.2) and (2.3) corresponds to an expansion wave going to the right, the lower sign to one going to the left.

It should be noted that the pressure behind a shock wave or an expansion wave, as well as Δu, for a fixed absolute value of H_{τ_0}, does not change with a change of sign in H_{τ_0}.

3. Conditions at vortex and contact discontinuities. At a vortex discontinuity [13], there are jumps only in $\mathbf{H}_\tau$ and $\mathbf{b}_\tau$, the tangential components of the magnetic field and the velocity; but their magnitudes remain unchanged. The change in the field and in the velocity are coupled by the relation

$$\mathbf{b}_{\tau_1} - \mathbf{b}_{\tau_0} = \mp (\mathbf{h}_1 - \mathbf{h}_0) V_0 \tag{3.1}$$

The upper sign corresponds to a wave going to the right, and the lower sign to one going to the left.

At a contact discontinuity [13]

$$\mathbf{H}_{\tau_1} = \mathbf{H}_{\tau_0}, \qquad p_1 = p_0, \qquad \mathbf{b}_{\tau_1} = \mathbf{b}_{\tau_0} \tag{3.2}$$

The density and the other remaining thermodynamic variables may undergo a jump.

In what follows, the case of a tangential discontinuity ($H_n = 0$) will not be considered, since, for all discontinuities the normal component of the magnetic field is continuous, while the case of the resolution of an arbitrary discontinuity, when the field at $t = 0$ is parallel to the surface of the discontinuity, reduces to a case of pure gasdynamics [5].

We will first consider the plane problem of the resolution of an arbitrary discontinuity, when $w_0 = w_0' = H_{z_0} = H_{z_0}' = 0$.

4. Combination of two waves and a contact discontinuity. Let us investigate the possibility that an arbitrary discontinuity be resolved into two shock waves or self-similar waves separated by a contact discontinuity. At the contact discontinuity Equations (3.2) must be satisfied. We shall see whether these conditions can be satisfied by any two shock waves or self-similar waves.

Let us investigate the possibility of the combinations R^-KR^- and R^+KR^+. From the conditions at a contact discontinuity, (3.2), and from (2.1), it follows that $(q_{\pm 1} - q_{\pm 1}')(P_1 q_{\pm 1} q_{\pm 1} - 1) = 0$. It is not difficult to see that $P_1 q_{-1} q_{-1}' < 1$ in an R^--wave and $P_1 q_{+1} q_{+1}' > 1$ in an R^+-wave. Therefore $q_{+1} = q_{+1}'$. This means that the R^-KR^-- and R^+KR^+-combinations are possible only in the case where the point p_0', H_{y_0}' lies on the curve relating p and H_y in R^-- and R^+-waves, respectively, and goes through the point p_0, H_{y_0}.

It is also clear that these combinations are mutually exclusive. When $p_0 > p_0'$, $|H_{y_0}| > |H_{y_0}'|$, the R^+KR^+-combination is possible; when $p_0 > p_0'$, $|H_{y_0}| < |H_{y_0}'|$, the R^-KR^--combination is possible.

In what follows we shall assume for definiteness that

$$p_0 > p_0', \qquad |H_{y_0}| > |H_{y_0}'|, \qquad H_{y_0} H_{y_0}' > 0.$$

Let us examine the curves giving the relation between p and H_y in S^+-, S^--, R^+-, R^--waves (Fig. 2), which are described by Equations (1.2) and (2.1), respectively. The dotten line in Fig. 2 shows the possible form of the line corresponding to the S^+-wave. From an examination of these curves it follows that the combinations of two shock or self-similar waves and a contact surface that are possible are the following (Figs. 3 to 6):

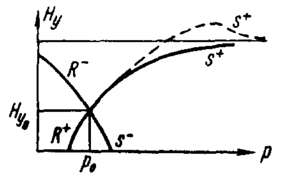


Fig. 2.

1) R^-KS^+, R^+KS^-, S^+KS^+, R^+R^-K, KS^-S^+ (4.1)

if $p_0 > p_+(p_0', H_{y_0}', H_y = H_{y_0})$, $H_{y_0}' > H_+(p_0, H_{y_0}, p = p_0')$

2) R^+KS^+, S^-KS^+, R^+KS^-, R^+R^-K, KR^-S^+ (4.2)

if $p_0 < p_+(p_0', H_{y_0}', H_y = H_{y_0})$, $H_{y_0}' > H_+(p_0, H_{y_0}, p = p_0')$

3) R^+KR^-, S^-KS^+, KR^-S^+, R^+S^-K (4.3)

if $p_0 < p_+(p_0', H_{y_0}', H_y = H_{y_0})$, $H_{y_0}' < H_+(p_0, H_{y_0}, p = p_0')$

4) R^-KS^+, S^+KS^+, R^+KR^-, R^+KS^+, KS^-S^+, R^+S^-K

if $p_0 > p_+(p_0', H_{y_0}', H_y = H_{y_0})$, $H_{y_0}' < H_+(p_0, H_{y_0}, p_0 = p_0')$ (4.4)

The S^-KS^- combination is possible if $p_-(p_0', H_{y_0}', H_y = 0) > p_-(p_0, H_{y_0}, H_y = 0)$.

In the case described by the inequalities (4.2) and (4.3) the point $p_+(p_0', H_{y_0}', H_y = H_{y_0})$ (Fig. 4), and also in the case described by inequalities (4.1) and (4.2), the point $H_+(p_0, H_{y_0}, p = p_0')$ (Figs. 3, 4), may even not exist.

Figs. 3 to 6 in the H_yp-plane, and Figs. 7 to 10 in the $\Delta u \Delta v$-plane apply to the cases defined by the inequalities (4.1) to (4.4), respectively.

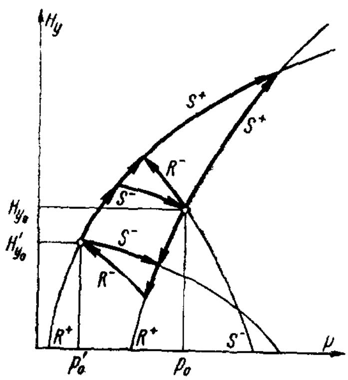


Fig. 3.

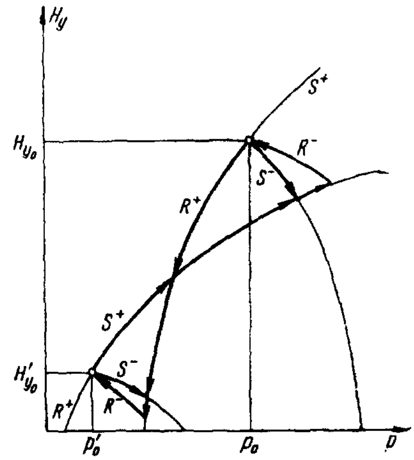


Fig. 4.

For definiteness, it has been assumed that $H_{y_0} > 0$, $H_{y_0}' > 0$.

If in the case described by inequalities (4.2) and (4.3) there is an S^+KS^+ combination, then in Figs. 4, 5, 8 and 9 there will be corresponding point. Cf. the remark at the end of Section 6.

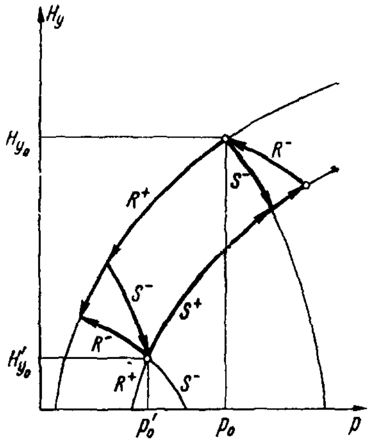


Fig. 5.

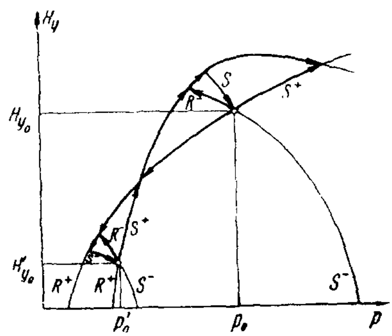


Fig. 6.

If $H_{y_0} < 0$, $H_{y_0}' < 0$, then Figs. 3 to 6 and 7 to 10 will not be changed if the values $-H_y$ and $-\Delta v$ are plotted on the vertical axis instead of the values H_y and Δv, respectively. The arrows indicate the direction of the change of quantities in S^+-, S^--, R^+- and R^--waves.

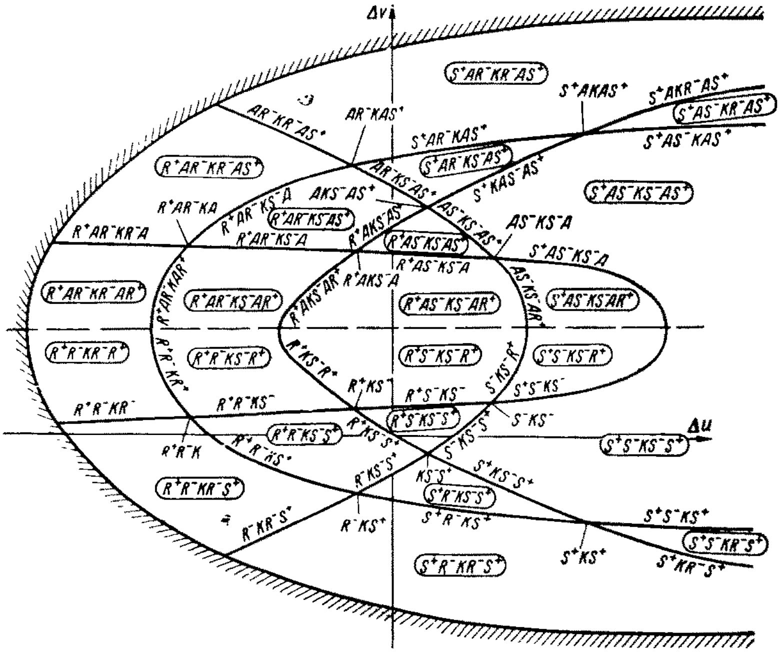


Fig. 7.

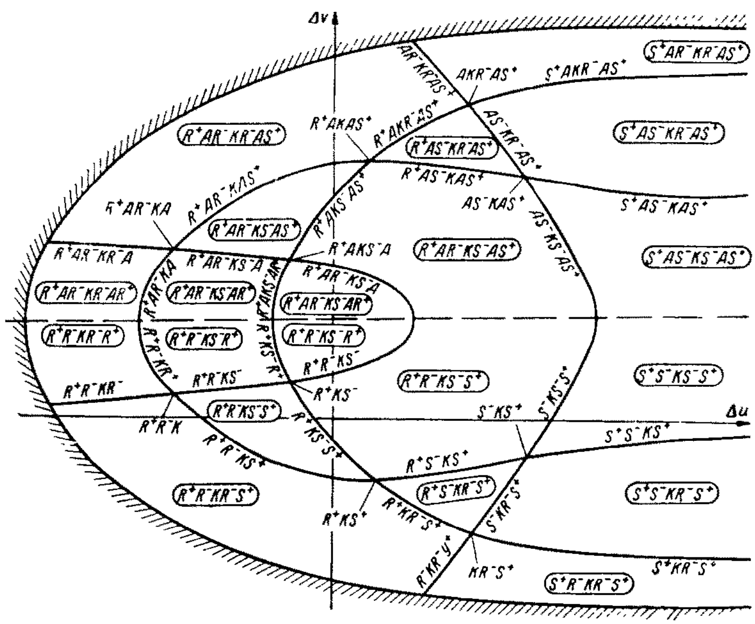


Fig. 8.

In all these cases, if a combination of two waves is possible, then specification of p_0, H_{y_0}, p_0', H_{y_0}' uniquely determines p_1, H_{y_1}, and, it follows, also Δu, Δv. In the $\Delta u \Delta v$-plane combinations of the type considered correspond to discrete points.

If the inequalities (4.1) to (4.4) become equalities, then the aggregate of combinations, corresponding to those equalities, becomes particular cases of those considered, as may be easily seen from the figures in the $H_y p$-plane.

5. Combinations consisting of three waves and a contact surface. We will now consider combinations consisting of three shock waves or self-similar waves and a contact discontinuity; we shall show that in the $\Delta u \Delta v$ -plane such a combination corresponds to a line.

In fact, in the $\Delta u \Delta v$ -plane, combinations of two waves correspond to points, i.e. Δu and Δv are constants depending only on p_0, H_{y_0}, p_0', H_{y_0}'. Addition of another wave means the addition of another term in the equation for Δu, Δv, and an additional parameter characterizing the strength of this wave. These one-parameter equations map a line in the $\Delta u \Delta v$ -plane.

We shall show how to construct these equations, for example the $R^- K R^- S^+$-combinations. From Equations (1.4), (1.5), (2.2) and (2.3) we have

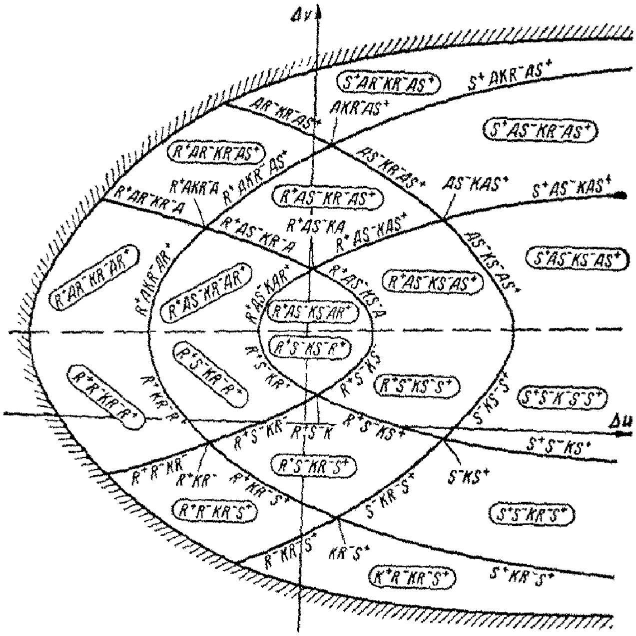
$$u_1 = u_0 + \psi_- = u_2' = u_1' - \psi_-' = u_0' + f_+' - \psi_-'$$
$$v_1 = v_0 + \chi_- = v_2' = v_1' - \chi_- = v_0' - \varphi_+' - \chi_-'$$

From this

$$\Delta u \equiv u_0 - u_0' = -\psi_- - \psi_-' + f_+', \quad \Delta v \equiv v_0 - v_0' = -\chi_- - \chi_-' - \varphi_+' \quad (5.1)$$

From the conditions at the contact discontinuity, $p_1' = p_1'(p_0, H_{y_0}, p_0', H_{y_0}')$, while $p_1 = p_2'$ in these equations is an independent parameter defining the strength of the R^- waves. If $p_0 = p_0'$ the strength of the rightward-propagating R^--wave is equal to zero. For this value of the parameter on the curve describing Equations (5.1), we find ourselves at the point which corresponds to the combination $R^- K S^+$.

For $p_1 = 0$ a point is obtained which corresponds to the combination $R^-_{max} K R^-_{max} S^+$ when the strength of the R^--waves is a maximum. In going across them a vacuum is attained. It is not difficult to see that, in


Δv
Δu
S+AR-KR-AS+
S+AKR-AS+
AR-KR-AS+
AKR-AS+
S+AS-KR-AS+
R+AR-KR-AS+
AS-KR-AS+
R+AKR-A
R+AKR-AS+
R+AS-KR-AS+
AS-KAS+
S+AS-KAS+
R+AR-KR-A
R+AS-KA
R+AS-KR-A
R+AS-KAS+
AS-KS-AS+
S+AS-KS-AS+
R+AR-KR-AR+
R+AKR-AR+
R+AS-KR-AR+
R+AS-KAR+
R+AS-KS-A
R+AS-KS-AS+
R+AS-KS-AR+
R+S-KS-R+
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S-KR-S+
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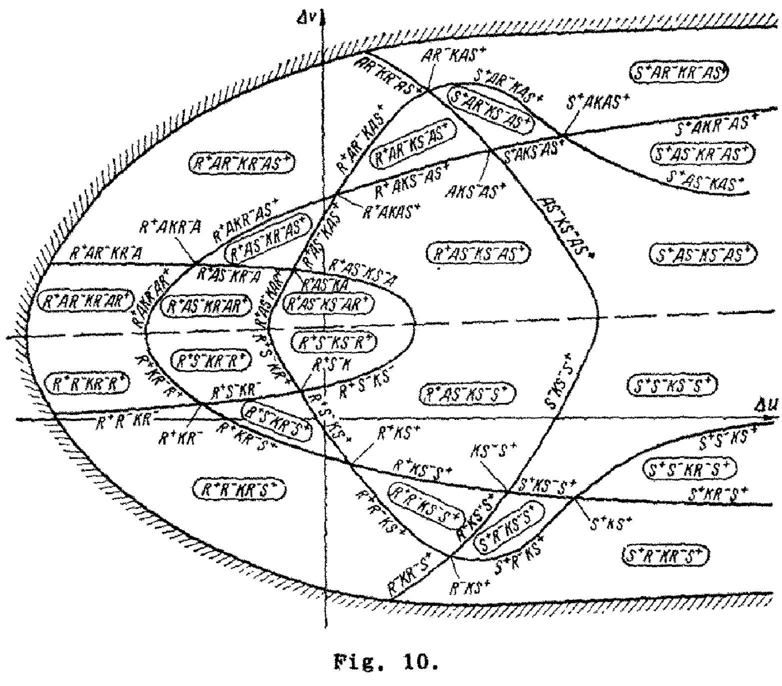
S+KR-S+

Fig. 9.


Δv
Δu
AR-KAS+
AR-KR-AS+
S+AR-KAS+
S+AR-KS-AS+
S+AKAS+
S+AR-KR-AS+
S+AKR-AS+
S+AS-KR-AS+
S+AS-KAS+
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R+AR-KR-A
R+AS-KR-A
R+AS-KS-A
R+AS-KA
R+AS-KS-AS+
AS-KS-AS+
S+AS-KS-AS+
R+AR-KR-AR+
R+AKR-AR+
R+AS-KR-AR+
R+AS-KAR+
R+AS-KS-AR+
R+S-KS-R+
R+S-KR-R+
R+S-KR-
R+S-K
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R+AS-KS-S+
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R+R-KR-S+
R+R-KS-S+
R+KS-S+
S+R-KS-S+
S+KS+
R+R-KS+
S+R-KS+
R-KR-S+
R-KS+
S+R-KR-S+

Fig. 10.

addition to the line $R^-KR^-S^+$, there terminate at the point R^-KS^+ the lines corresponding to the combinations $R^+R^-KS^+$, $S^+R^-KS^+$, $R^-KS^-S^+$, where the strength of the leftward-going R^+-, S^+-waves and of the S^--wave, respectively, are equal to zero.

From the equations of these lines, constructed similarly to Equation (5.1), it follows that they are arranged as shown on Fig. 7. Let us explain where the line $S^+R^-KS^+$, emerging from point R^-KS^+, ends. From Fig. 3 it is evident that in going along this line the strength of the R^--wave will decrease to zero. Thus the $S^+R^-KS^+$-line comes to the point corresponding to the S^+KS^+-combination. From similar arguments it follows that the lines $R^+R^-KS^+$ and $R^-KS^-S^+$ end at points which correspond to the combinations R^+R^-K and KS^-S^+, respectively. At each of those points three more lines can arrive, and so forth.

From the above it follows that combinations consisting of three shock or self-similar waves and a contact discontinuity correspond to lines in the $\Delta u \Delta v$-plane. The points at the intersections of the lines correspond to the combinations, considered earlier, consisting of two shock or self-similar waves and a contact discontinuity. Every such point separates one line from another. The lines may extend to infinity, for example the $S^+S^-KS^+$-, $S^+KR^-S^+$-lines on Fig. 7. Also, lines may terminate at points corresponding to maximum wave strengths. Thus, for example, it is evident from Figs. 3 and 7 that the line $R^+R^-KR^+$ continues until the strength of the R^+-waves is a maximum, and the lines $R^+KS^-R^+$, $S^-KS^-R^+$, $S^+KS^-S^+$ terminate at points where the strength of R^+-, S^--waves is maximum, that is, where the tangential component of the magnetic field behind the R^+- or S^--waves is equal to zero. Lines joining points corresponding to maximum strength of R^+- and S^--waves we will call dividing lines.

6. Combinations with a vortex discontinuity. Inasmuch as lines corresponding to certain combinations are continuations one of the other, there are (Figs. 7 to 10) four distinct lines corresponding to the combinations investigated, of three waves and a contact discontinuity. For instance, (Fig. 7) one line corresponds to the combinations $R^-KR^-S^+$, $R^-KS^-S^+$, $S^-KS^-S^+$, $S^-KS^-R^+$, another to $R^+R^-KR^-$, $R^+R^-KS^-$, $R^+S^-KS^-$, $S^+S^-KS^-$, etc. Examining the equations of these lines it is easy to see that the maximum ordinates of these lines are points on a dividing line.

Above those points our lines can be continued only by including vortex discontinuities in the combinations considered. Since the investigation is for the case $H_{y_0}H_{y_0}' > 0$, a vortex discontinuity must exist on both sides of the contact discontinuity or be absent entirely. In the plane case being investigated, a vortex discontinuity rotates the tangential component of magnetic field 180°, and, while changing the tangential component of velocity, leaves the remaining parameters unchanged. For every line or point lying below a dividing line there is a line or point above the dividing line which corresponds to a combination of the same waves as for the lower point plus two vortex discontinuities.

Consider, for example, the line which corresponds to the $R^+R^-KR^+$-combination (Fig. 7). Its equation, constructed like Equation (5.1), will be

$$\Delta u \equiv u_0 - u_0' = -\psi_- - \psi_+' - \psi_-, \quad \Delta v \equiv v_0 - v_0' = \chi_+ + \chi_+' - \chi_-$$

Let us construct the equations of the line which corresponds to the $R^+AR^-KAR^+$-combination. From (2.2), (2.3) and (3.1) it follows that

$$\Delta u = -\psi_+ - \psi_+' - \psi_-, \qquad \Delta v = \chi_+ + \chi_+' + \chi_- + 2h_1V_1 + 2h_1'V_1'$$

As is evident from an examination of the equations constructed, for given initial parameters Δu is the same on both lines, since the pressure and the absolute value of the field in the region between the waves do not change, while Δv differs by the quantity

$$2(\chi_- + h_1V_1 + h_1'V_1')$$

It can be shown that this sum is the distance between the lines $R^+R^-KS^+$ and $R^+AR^-KAS^+$, $S^+R^-KS^+$ and $S^+AR^-KAS^+$, $S^+KR^-S^+$ and $S^+AKR^-AS^+$, except of course that χ_-, h_1, h_1', V_1, V_1' will be different. The distance between the points S^+KS^+ and S^+AKAS^+ is equal to $2h_1(V_1 + V_1')$, since $\chi_- = 0$ and $h_1 = h_1'$.

We will show that the lines $R^+R^-KR^-$ and $R^+AR^-KAR^+$ come to a common point on a dividing line, that is, the distance in the Δv-direction between lines is equal to zero on the dividing line.

From the relations for expansion waves it follows that:

1) if $H_\tau = 0$ in an R^+-wave, then $P < 1$ at that point;

2) if $P < 1$ and $H_\tau = 0$ ahead of an R^--wave, then this wave is purely a gasdynamic one, i.e. the jump in the tangential component of velocity is equal to zero. Now on the dividing line after the R^+-wave going leftward the tangential component of the field is equal to zero ($H_{\tau_1} = 0$), that means $P < 1$. Consequently, behind this R^+-wave comes a gasdynamic expansion wave in which $\chi_- = 0$, from which it follows that in the rightward-moving R^+-wave H_τ must also change to zero ($H_{\tau_1}' = 0$), i.e. the lines $R^+R^-KR^+$ and $R^+AR^-KAR^+$ come to one point on the dividing line.

It can be shown that on the dividing line the coordinates of the lines

$R^+KS^-R^+$ and $R^+AKS^-AR^+$ coincide, etc. (Fig. 7), and also those of the corresponding lines on the other figures.

We note that in the $\Delta u \Delta v$-plane there may be, generally speaking, two or more of the points S^+KS^+, S^+AKAS^+, S^-KS^-, AS^-KS^-A, and thus of the lines $S^+KS^-S^+$, $S^+R^-KS^+$, etc., if the lines S^+, S^- emerging from the points p_0, H_{y_0}, p_0', H_{y_0}', respectively, have two or more points of intersection with each other.

The question of intersections of S^+- and S^--lines in the $H_y p$-plane was not investigated in general.

7. Combinations of four waves and a contact discontinuity. It is not difficult to see that the combination consisting of four waves and a contact discontinuity corresponds to a region in the $\Delta u \Delta v$-plane, since each combination of four waves and a contact discontinuity corresponds to a system of two equations for Δu and Δv, depending on two parameters; the lines investigated above are the boundaries of these regions.

Analogously, every combination of four shock or self-similar waves, two vortex discontinuities and a contact discontinuity also corresponds to a region in the $\Delta u \Delta v$-plane, since the addition of a plane vortex discontinuity does not introduce any new parameter. The boundaries of such regions are lines which correspond to combinations of three shock or self-similar waves, two vortex discontinuities and a contact discontinuity, investigated in the preceding article. The equations of such lines are constructed quite analogously to Equation (5.1).

8. The dividing line. The dividing line separates the regions $R^+R^-KS^-R^+$ and $R^+AR^-KS^-AR^+$, $R^+S^-KS^-R^+$ and $R^+AS^-KS^-AR^+$, etc., and is shown in Figs. 7 to 10 by a dotted line. It is convenient to write the equation of the dividing line separately for each pair of regions which are separated by it.

Let us write the equation of the portion of the dividing line between the regions $R^+R^-KR^-R^+$ and $R^+AR^-KR^-AR^+$: In the region $R^+R^-KR^-R^+$, Δu and Δv satisfy the equations

$$\Delta u = -\psi_+ - \psi_+' - \psi_- - \psi_-', \qquad \Delta v = \chi_+ + \chi_+' - \chi_- - \chi_-' \qquad (8.1)$$

which are obtained analogously with Equations (5.1). On the dividing line the strength of the R^+-wave is a maximum; the R^--wave is a purely gasdynamical one; therefore p_1 is expressed in terms of p_0, H_{y_0}, and p_1' is expressed in terms of p_0', H_{y_0}'; $p_2 = p_2'$ (an independent parameter which changes from zero (the point of intersection of the dividing line with the vacuum line) to p_1' (the point of intersection of the dividing line with the lines $R^+R^-KR^+$ and $R^+AR^-KAR^+$)).

From the preceding it follows that the system

$$\Delta u = -\psi_{+\max} - \psi_{+\max}' - \psi_- - \psi_-', \qquad \Delta v = \chi_{+\max} + \chi_{+\max} \quad (8.2)$$

is obtained from (8.1); it is the system of one-parameter equations of the portion of the dividing line under consideration. It is clear that this portion of the dividing line is straight.

The equations of the other portions of the dividing line are constructed in exactly the same way; they will not be straight lines.

9. The vacuum line. Points on the vacuum line, as well as points lying beyond the vacuum line, correspond to combinations containing two R^--waves of maximum strength. Going through them, a vacuum is obtained. Let us write the equation of the portion of the vacuum line which bounds the region corresponding to the $R^+R^-KR^-R^+$-combination. Δu and Δv in this region satisfy Equations (8.1). In these equations, $p_2 = 0$, $p_1' = p_1'(p_0, H_{y_0}, p_0', H_{y_0}', p_1)$ on the vacuum line (Fig. 7). We again obtain a one-parameter family of equations with the independent parameter p_1, which will also be the equation of the portion of the vacuum line under consideration.

The equations of the other portions are obtained analogously. The vacuum line is bounded on the left, but extends to (upper and lower) infinity on the right. In Figs. 7 to 10 the vacuum line is distinguished by cross-hatching.

10. The case $p_0 > p_0'$, $|H_{y_0}| < |H_{y_0}'|$. As before, let $H_{y_0}H_{y_0}' > 0$. From an examination of the curves which depict the relations between H_y and p in S^+-, S^--, R^+-, and R^--waves (Figs. 11 to 14), it follows that in this case the following combinations of two shock or self-similar waves and a contact surface are possible:

1) R^-KS^+, R^+KS^-, KS^-S^+, R^+R^-K

if $\quad p_0 > p_-'$, $(p_0', H_{y0}'. H_y = H_{y0})$, $\quad H_{y0}' < H_-(p_0, H_{y0}, p = p_0')$ (10.1)

2) R^-KS^-, R^-KS^+, S^+KS^-, R^+R^-K, KS^-R^+

if $\quad p_0 < p_-(p_0', H_{y0}', H_y = H_{y0})$, $\quad H_{y0}' < H_-(p_0, H_{y0}, p = p_0')$ (10.2)

$$3)\quad S^+KS^-,\quad R^-KR^+,\quad KS^-S^+,\quad S^+R^-K$$

if $\quad p_0 < p_-(p_0', H_{y0}', H_y = H_{y0}), \qquad H_{y0}' > H_-(p_0, H_{y0}, p = p_0')$ (10.3)

$$4)\quad R^-KS^-,\quad R^+KS^-,\quad R^-KR^+,\quad KS^-S^+,\quad S^+R^-K$$

if $\quad p_0 > p_-(p_0', H_{y0}', H_y = H_{y0}), \qquad H_{y0}' > H_-(p_0, H_{y0}, p = p_0')$ (10.4)

The S^-KS^--combination is possible,

if
$$p_-(p_0', H_{y0}', H_y = 0) < p_-(p_0, H_{y0}, H_y = 0), \qquad H_{y0} < H_-(p_0', H_{y0}', p = p_0)$$

Figures 11 to 14 and 15 to 18 correspond to these four cases, respectively. We note that qualitatively Figs. 7 and 15 differ from each other only in that on Fig. 15 the points corresponding to the combinations S^+KS^- and S^-KS^- are missing.

The lines and regions on these drawings are constructed exactly as for the case $p_0 > p_0'$, $|H_{y_0}| > |H_{y_0}'|$, $H_{y_0}H_{y_0}' > 0$.

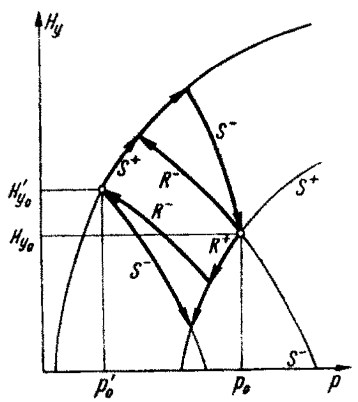

Fig. 11.

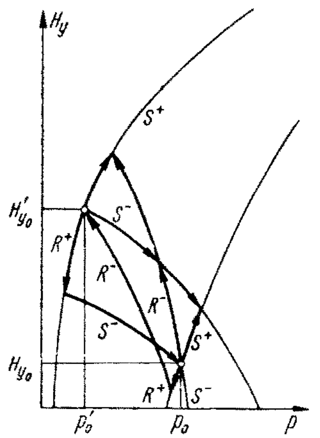

Fig. 12.

If the combination S^+KS^- occurs in the cases described by Equations (10.1) to (10.4), then in Figs. 11 to 14 and 15 to 18 there will be a corresponding point (cf. the remarks at the end of Section 6).

If the inequalities (10.1) to (10.4) become equalities the set of combinations corresponding to these equalities are particular cases of the corresponding set of combinations under consideration, as is easily seen from Figs. 11 to 14 in the $H_y p$-plane.

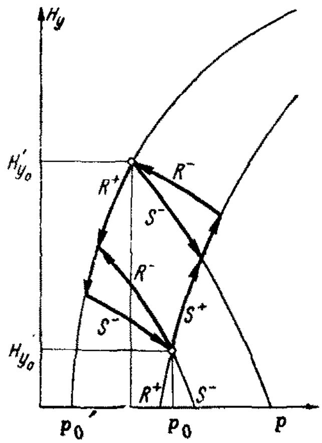

Fig. 13

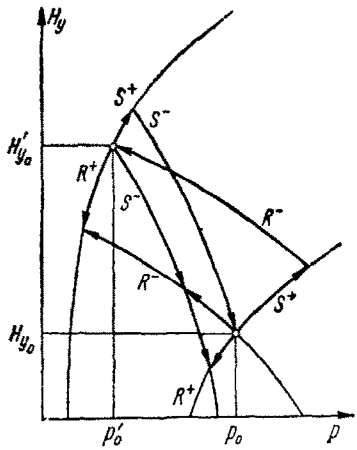

Fig. 14.

11. The case $p_0 > p_0'$, $|H_{y_0}| > |H_{y_0}'|$, $H_{y_0}H_{y_0}' < 0$. For definiteness, let $H_{y_0} > 0$, $H_{y_0}' < 0$. In this case, in every combination there must be a vortex discontinuity going to left or right.

This is made necessary by the fact that neither a shock wave nor a self-similar wave can change the sign of the field, which is however necessary, since at the contact discontinuity the tangential components of the field must be equal. It is possible to investigate in detail the possibility of existence of combinations of two, three and more waves, repeating almost entirely the previous discussions. But even without a detailed investigation it is clear that to every combination of two waves and a contact surface, for the case $p_0 > p_0'$, $|H_{y_0}| > |H_{y_0}'|$, $H_{y_0}H_{y_0}' > 0$, there corresponds a combination consisting of the same waves, a vortex discontinuity and a contact discontinuity.

Here the pressure and the normal velocity components between the respective waves and discontinuities must be equal in these combinations, since the quantities mentioned do not change if the sign of H_τ ahead of the wave is changed (the other parameters at the wave front remaining unchanged). The tangential component of velocity changes sign.

The same may be said about the lines and the regions. Thus it is possible qualitatively to redraw Figs. 7 to 10, replacing combinations without a vortex discontinuity or with two vortex discontinuities by the corresponding combination with one vortex discontinuity, going leftward or rightward. The dividing line in a given case separates regions in

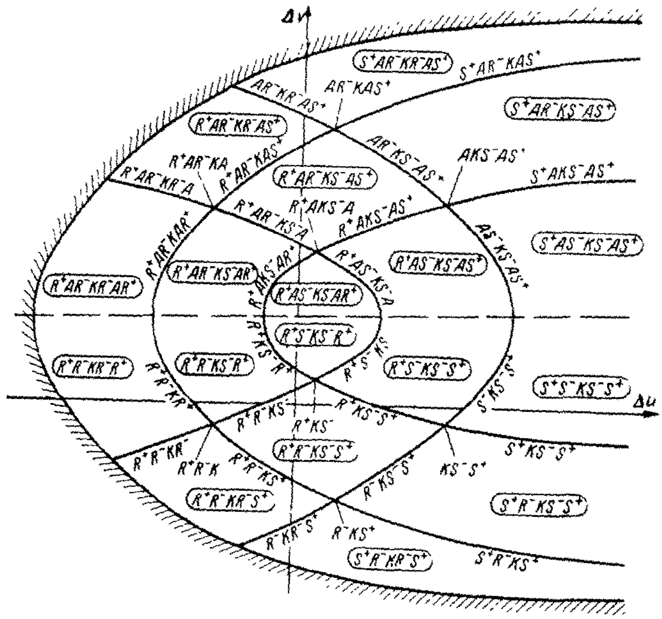

Fig. 15.

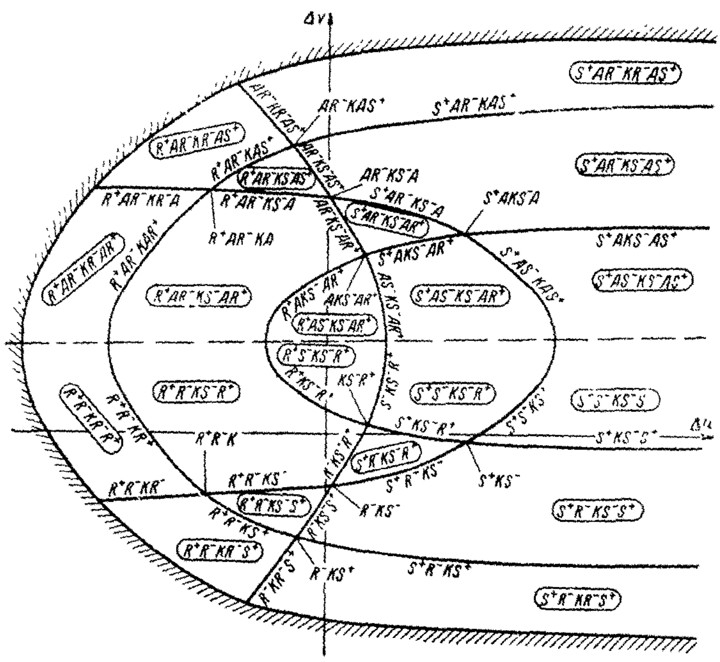

Fig. 16.

which the vortex discontinuity is going rightward from regions with a vortex discontinuity going leftward.

Let us clarify which regions are situated above the dividing line and which below. Let us investigate the combinations AR^-KS^+ and R^-KAS^+;

$$\Delta v = f_+' + \chi_- + 2h_0V_0 \qquad \text{for the first}$$

$$\Delta v \equiv \Delta v^* = f_+' - \chi_- - 2h_1'V_0' \qquad \text{for the second}$$

The difference $\Delta v - \Delta v^*$ is the same as in the case corresponding to $H_{y_0}H_{y_0}' > 0$. Therefore regions corresponding to combinations in which a rightward-going vortex discontinuity occurs are situated below the dividing line, regions with a leftward-propagating discontinuity are above.

Analogously, the diagrams for the case $p_0 > p_0'$, $|H_{y_0}| < |H_{y_0}'|$, $H_{y_0}H_{y_0}' < 0$ may be obtained qualitatively from Figs. 15 to 18 for the case $p_0 > p_0'$, $|H_{y_0}| < |H_{y_0}'|$, $H_{y_0}H_{y_0}' > 0$, by replacing combinations without a vortex discontinuity and with two vortex discontinuities by combinations with one vortex discontinuity, going rightward or leftward.

If $H_{y_0} < 0$, $H_{y_0}' > 0$ in the last two cases discussed, then the diagrams do not change if $v_0' - v_0$ is plotted on the ordinate instead of $v_0 - v_0'$.

12. Three-dimensional case of the problem. Let us investigate the three-dimensional problem of the resolution of an arbitrary discontinuity. The velocity and magnetic field vectors on both sides of the plane of the discontinuity lie in different planes. Therefore the conditions at the contact discontinuity cannot be satisfied without introducing three-dimensional vortex discontinuities.

Let us assume that the initial conditions are such that the three-dimensional initial disturbance resolves itself into an $R^+AR^-KAR^+$-combination of waves and discontinuities.

The equation for Δu will be the same as the equation for the same combination in the plane case. Let us construct the equation for $\Delta \mathbf{b}_\tau$

$$\mathbf{b}_{\tau 3} = \mathbf{b}_{\tau 2} + \frac{\mathbf{h}_2}{|\mathbf{h}_2|}\chi_- = \mathbf{b}_{\tau 1} + (\mathbf{h}_2 - \mathbf{h}_1)V_1 + \frac{\mathbf{h}_2}{|\mathbf{h}_2|}\chi_- = \mathbf{b}_{\tau 0} - \frac{\mathbf{h}_0}{|\mathbf{h}_0|}\chi_+ +$$

$$+ (\mathbf{h}_2 - \mathbf{h}_1)V_1 + \frac{\mathbf{h}_2}{|\mathbf{h}_2|}\chi_- = \mathbf{b}_{\tau 3}' = \mathbf{b}_{\tau 0}' + \frac{\mathbf{h}_0'}{|\mathbf{h}_0'|}\chi_+' - (\mathbf{h}_2' - \mathbf{h}_1')V_1'$$

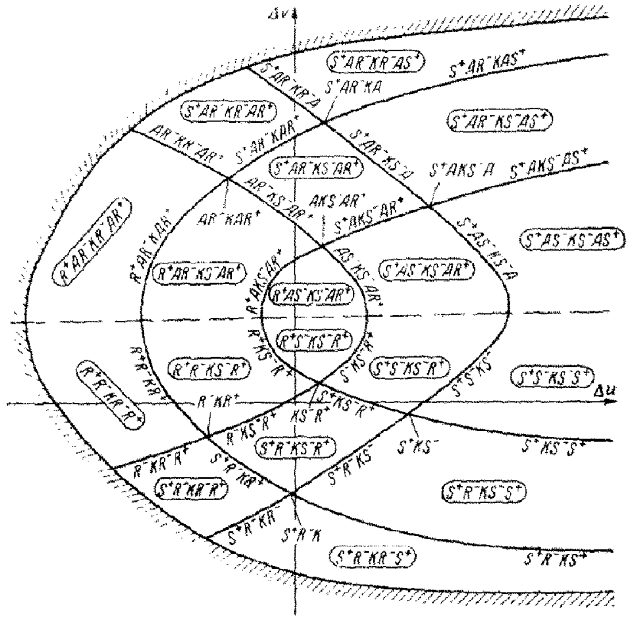

Fig. 17.

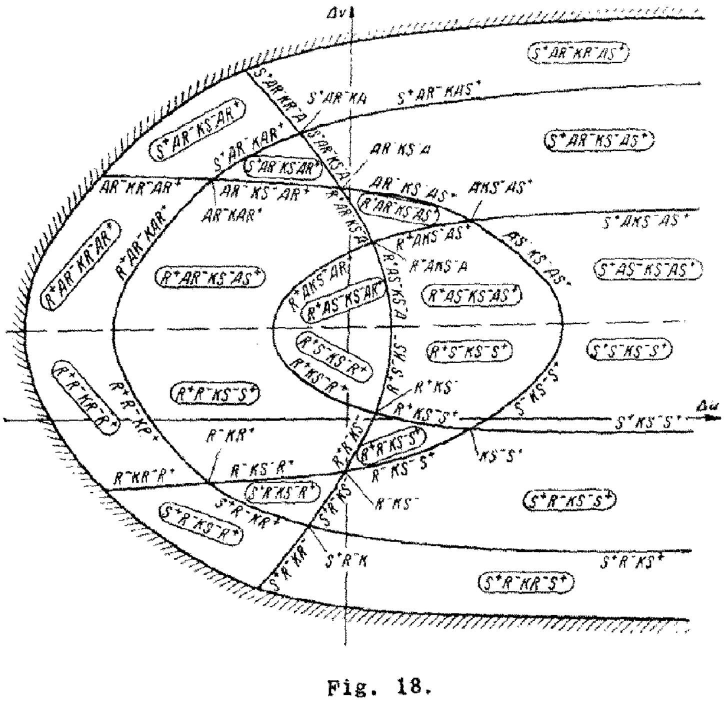

Fig. 18.

From this

$$(\mathbf{b}_{\tau 0} - \mathbf{b}_{\tau 0}' - \mathbf{L})^2 = \mathbf{R}^2 \tag{12.1}$$

where

$$\mathbf{L} = \frac{\mathbf{h}_0}{|\mathbf{h}_0|}\chi_+ + \frac{\mathbf{h}_0'}{|\mathbf{h}_0'|}\chi_+' + \mathbf{h}_1 V_1 + \mathbf{h}_1' V_1', \quad \mathbf{R} = -\mathbf{h}_2 V_1 - \mathbf{h}_2' V_1' - \frac{\mathbf{h}_2}{|\mathbf{h}_2|}\chi_-$$

which is the equation of a region with center at the point (Δu, $\Delta v = L_y$, $\Delta w = L_z$) and radius equal to $|\mathbf{R}|$.

Let $\mathbf{H}_{\tau 0} \parallel \mathbf{H}_{\tau 0}'$, and choose this direction for the y-axis, with the z-axis perpendicular to the y-axis and normal to the discontinuity surface. Then $L_z = 0$, i.e. the center of the region lies in the $\Delta u \Delta v$-plane. In this plane, Equation (12.1) will give two values of Δv: one of them lies on the line corresponding to the $R^+R^-KR^+$-combination; the other on the line corresponding to the $R^+AR^-KAR^+$-combination, in which the field is turned through 180° at the A-discontinuities; both values of Δv are at a distance $|\mathbf{R}|$ from the point with coordinates Δu, $\Delta v = L_y$. It can be shown that this point lies on the dividing line. Δu has one value at these three points.

Thus, it has been shown that for $\mathbf{H}_{\tau 0} \parallel \mathbf{H}_{\tau 0}'$ the surface corresponding to the $R^+AR^-KAR^+$-combination is obtained by rotating the line in the $\Delta u \Delta v$-plane which corresponds to the $R^+R^-KR^+$-combination around the dividing line. The intersection of this surface with the plane $\Delta w = 0$ will give two lines: one corresponds to the $R^+R^-KR^+$-combination, the other to the $R^+AR^-KAR^+$-combination, where the A-discontinuity rotates the field through 180°; this surface separates the two regions, $R^+AR^-KR^-AR^+$ and $R^+AR^-KS^-AR^+$.

All the remaining portions of the three-dimensional diagram for the general case of the resolution of an arbitrary discontinuity, with $\mathbf{H}_{\tau 0} \parallel \mathbf{H}_{\tau 0}'$, are constructed in an analogous way. In the same way, rotation of the vacuum line, lying in the $\Delta u \Delta v$-plane, around the dividing line will give the vacuum surface.

If $\mathbf{H}_{\tau 0}$ is not parallel to $\mathbf{H}_{\tau 0}'$, then $L_z \neq 0$ and the radius of the region does not change. Thus, in this case, the surface in the $\Delta u \Delta v \Delta w$-space which corresponds to the combination $R^+AR^-KAR^+$ is the surface which corresponds to the same combination of waves constructed for $\mathbf{H}_{\tau 0} \parallel \mathbf{H}_{\tau 0}'$ and displaced in accordance with Equation (12.1). From this it is clear that the criteria (4.1) to (4.4), (10.1) to (10.4), defining the aggregate of combinations for the case $\mathbf{H}_{\tau 0}$ not parallel to $\mathbf{H}_{\tau 0}'$, remain the same as in the case where $\mathbf{H}_{\tau 0} \parallel \mathbf{H}_{\tau 0}'$.

13. Conclusion. Now let the values of the parameters be given on both sides of the plane of the discontinuity. We shall show how the results of the solution may be used. Since we know $\mathbf{H}_{\tau 0}$, $\mathbf{H}_{\tau 0}'$, p_0, p_0', we know which of the necessary discontinuities obtained in this paper they satisfy, that is, we know which equations of the lines we must write, using Equations (1.4), (1.5), (2.2) and (2.3), in order to construct the corresponding drawing. After the drawing is constructed, and since Δu, Δv, Δw are known, we ascertain in which region the point with these coordinates lies, that is, we determine the combination of waves and discontinuities into which the initial discontinuity resolves itself. Equating the sum of the jumps of each magnetohydrodynamic quantity on each of the resulting waves and discontinuities to that of the initial jump, we obtain a system of algebraic equations which has to be solved numerically.

If the point Δu, Δv, Δw lies in the space beyond the vacuum line, additional investigation is needed. The vacuum appears behind maximum strength R^--waves, propagating in both directions. On the boundary between the vacuum and the medium the following equations [4] are satisfied:

$$p = 0, \qquad [\mathbf{H}_\tau] = 0, \qquad [\mathbf{E}_\tau] = 0$$

but in view of the infinite conductivity of the medium

$$\mathbf{E}_{\tau_3} = -[\mathbf{b}_3\mathbf{H}_3]_\tau, \qquad \mathbf{E}_{\tau_3}' = -[\mathbf{b}_3'\mathbf{H}_3']_\tau, \qquad \mathbf{E}_{\tau\,\mathrm{vac}} = \mathrm{const}, \ \mathbf{H}_{\tau\,\mathrm{vac}} = \mathrm{const}$$

since, in the self-similar problem, electromagnetic waves cannot exist in the vacuum region, that is

$$[(\mathbf{b}_3 - \mathbf{b}_3')\,\mathbf{H}_3]_\tau = 0 \tag{13.1}$$

Let the resolution be a plane one, and suppose we are at some point on the portion of the vacuum line which bounds the region corresponding to the combination $R^+R^-KR^-R^+$. Then

$$\begin{aligned} u_2 &= u_0 + \psi_+ + \psi_-, \qquad & u_2' &= u_0' - \psi_+' - \psi_-' \\ v_2 &= v_0 - \chi_+ + \chi_-, \qquad & v_2' &= v_0' + \chi_+' - \chi_-' \end{aligned} \tag{13.2}$$

Writing out the vector product and putting into it the expressions for u_2, u_2', v_2, v_2' from (13.2) we obtain

$$\begin{aligned} &[(u_0 - u_0') + (\psi_+ - \psi_+' + \psi_- - \psi_-')]\,H_{y_2} - \\ &-[(v_0 - v_0') + (-\chi_+ - \chi_+' + \chi_- + \chi_-')]\,H_x = 0 \end{aligned} \tag{13.3}$$

In the $\Delta u \Delta v$-plane Equation (13.3) is a straight line.

The equations of the straight lines emerging from other portions of the vacuum line are constructed in an analogous way.

If a piston speed lies beyond the vacuum line, then from that point to the vacuum line one must go along a straight line of the type (13.3). At every point of the vacuum line the combination into which the initial disturbance resolves itself is known.

If the problem is three-dimensional, then Equations (13.1) are equations of straight lines in the $\Delta u \Delta v \Delta w$-space, filling, to the left of the vacuum line, all the space which can be obtained by rotating the corresponding straight lines lying in the $\Delta u \Delta v$-plane by the method outlined in Section 12.

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